Ergodic Theory - Week 9

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1 Classifying measure preserving systems

P1. **(a)** Show that if the system $(X, \mathcal{B}, \mu, T)$ is mixing, then for all strictly increasing sequences of integers n_k and any $A \in \mathcal{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) = 1.$$

> Let $B = \bigcup_{k=1}^{+\infty} T^{-n_k}A$. Since the system is mixing, we have that
>
> $$\lim_{n\to+\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$
>
> In particular, we also have
>
> $$\lim_{k\to+\infty} \mu(T^{-n_k}A \cap B) = \mu(A)\mu(B).$$
>
> Let $\varepsilon > 0$. If k is sufficiently large, we have that
>
> $$\mu(T^{-n_k}A \cap B) \le \mu(A)\mu(B) + \varepsilon. \tag{1}$$
>
> However, note that
>
> $$T^{-n_k}A \cap B = \bigcup_{\ell=1}^{+\infty} (T^{-n_k}A \cap T^{-n_\ell}A) \supseteq T^{-n_k}A \cap T^{-n_k}A.$$
>
> We infer that
>
> $$\mu(T^{-n_k}A \cap B) \ge \mu(T^{-n_k}A) = \mu(A).$$
>
> Combining this with (1), we get
>
> $$\mu(A) \le \mu(A)\mu(B) + \varepsilon \implies \mu(B) \ge \frac{\mu(A) - \varepsilon}{\mu(A)}.$$
>
> Taking ε sufficiently small, we conclude that $\mu(B) = 1$.

(b) Show that if the system $(X, \mathcal{B}, \mu, T)$ is weak-mixing, then for all sequences of positive integers n_k with positive density and any $A \in \mathcal{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k}A\right) = 1.$$

* Show that the converse holds as well.
Hint: For the converse, use the fact that if a system has an eigenfunction, then it has a factor map to a rotation system.

Let B be defined as above. Since the system is weak-mixing there exists a set $E \subseteq \mathbb{N}$, such that E^c has zero density and

$$\lim_{n\to+\infty, n\in E} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

We observe that E must contain infinitely many elements of the sequence n_k, as otherwise, the set E^c would contain all elements $n_K, n_{K+1}, \ldots$ for some $K \in \mathbb{N}$ and this would imply that E^c cannot have zero density.
Let n_{N_k} be a subsequence of n_k such that $n_{N_k} \in E$ for all $k \in \mathbb{N}$. We deduce that

$$\lim_{k\to+\infty} \mu(T^{-n_{N_k}}A \cap B) = \mu(A)\mu(B).$$

Let $\varepsilon > 0$. If k is sufficiently large, we have that

$$\mu(T^{-n_{N_k}}A \cap B) \leq \mu(A)\mu(B) + \varepsilon.$$

However, note that

$$T^{-n_{N_k}}A \cap B = \bigcup_{\ell=1}^{+\infty}(T^{-n_{N_k}}A \cap T^{-n_\ell}A) \supseteq T^{-n_{N_k}}A \cap T^{-n_{N_k}}A.$$

We infer that

$$\mu(T^{-n_{N_k}}A \cap B) \geq \mu(T^{-n_{N_k}}A) = \mu(A).$$

Similarly to part (a), we get

$$\mu(A) \leq \mu(A)\mu(B) + \varepsilon \implies \mu(B) \geq \frac{\mu(A) - \varepsilon}{\mu(A)}.$$

Taking ε sufficiently small, we conclude that $\mu(B) = 1$.
For the converse, we assume that for every set A of positive measure and any sequence n_k of positive density, we have $\mu(\cup_{k=1}^{+\infty} T^{-n_k}A) = 1$. We suppose that the system $(X, \mathcal{B}, \mu, T)$ is not weak-mixing and we will arrive at a contradiction.
For simplicity, let us say that a system $(X, \mathcal{B}, \mu, T)$ is "good" if for every set $A \in \mathcal{B}$ of positive measure and any sequence n_k of positive density, we have $\mu(\cup_{k=1}^{+\infty} T^{-n_k}A) = 1$.
The proof is a bit lengthy, but the strategy is as follows: if the system is not weak-mixing, it contains a factor that is isomorphic to a rotation on a compact abelian group. We show that the property of being good descends to this factor and then the problem reduces to showing that the rotation system does not possess the "good" property.
We prove the following claim:
Claim: A factor of a good system is also a good system.

Proof. To prove this claim, assume that we have a factor map $\pi : (X, \mathcal{B}, \mu, T) \to (Y, \mathcal{A}, \nu, S)$ where the first system is good. Let $B \in \mathcal{A}$ have positive measure and let n_k be any sequence with positive density. We want to show that

$$\nu\left(\bigcup_{k=1}^{+\infty} S^{-n_k}B\right) = 1.$$

We prove this by showing that

$$\nu\left(\bigcup_{k=1}^{N} S^{-n_k}B\right) = \mu\left(\bigcup_{k=1}^{N} T^{-n_k}A\right) \tag{2}$$

for all $N \in \mathbb{N}$, where $A = \pi^{-1}(B) \in \mathcal{B}$. Indeed, since π is a factor map, we have $\mu(A) = \mu(\pi^{-1}B) = \nu(B) > 0$. Then, taking the limit as $N \to +\infty$ and using the fact that the system $(X, \mathcal{B}, \mu, T)$ is good, we reach the desired conclusion.

In order to show (2), it suffices to prove that

$$\pi^{-1}\left(\bigcup_{k=1}^{N} S^{-n_k}B\right) = \bigcup_{k=1}^{N} T^{-n_k}A$$

and the claim would follow from the fact that ν is the pushforward of μ under π.

Let $x \in \bigcup_{k=1}^{N} T^{-n_k}A$ (x is a point on X). Then, we have that $T^{n_k}x \in A$ for some $k \in \{1, \ldots, N\}$. Since π is a factor map, we have $\pi(T^{n_k}(x)) = S^{n_k}(\pi(x))$, so that $S^{n_k}(\pi(x)) \in \pi(A)$. As $A = \pi^{-1}(B)$, we deduce that $S^{n_k}(\pi(x)) = \pi(T^{n_k}x) \in B$. Therefore, $\pi(x) \in S^{-n_k}B \subseteq \bigcup_{k=1}^{N} S^{-n_k}B$, so that $x \in \pi^{-1}(\bigcup_{k=1}^{N} S^{-n_k}B)$.

Conversely, assume that $x \in \pi^{-1}\left(\bigcup_{k=1}^{N} S^{-n_k}B\right)$. Thus, $\pi(x) \in \bigcup_{k=1}^{N} S^{-n_k}B$ and we infer that there exists $k \in \{1, \ldots, N\}$ such that $S^{n_k}\pi(x) \in B$. We deduce that $\pi(T^{n_k}x) \in B$ and thus, $T^{n_k}x \in \pi^{-1}(B) = A$. We conclude that $x \in \bigcup_{k=1}^{N} T^{-n_k}A$.

Combining the last two arguments, we deduce that

$$\pi^{-1}\left(\bigcup_{k=1}^{N} S^{-n_k}B\right) = \bigcup_{k=1}^{N} T^{-n_k}A$$

and the claim follows. □

We return to our exercise. Since the system is not weak-mixing, there exists an eigenfunction f with eigenvalue $e(a)$ for some $a \in [0, 1)$. Using exercise 4 a) from exercise sheet 7, we know that there exists a factor map $(X, \mathcal{B}, \mu, T) \to (Y, \mathcal{A}, \nu, S)$, such that the system $(Y, \mathcal{A}, \nu, S)$ is a rotation on a compact abelian group. In fact, an inspection of the argument in this exercise shows that we can take Y to the rotation by a on the torus $\mathbb{T}$, when a is irrational, or the rotation on q points, when $a = p/q, (p, q) = 1$ is rational. We conclude the exercise by showing that the last two types of systems are not good.

We start with the rotation on q points. Namely, we let $Y = \{0, \ldots, q-1\}$ with the uniform measure and the map $Sx = x + 1 \pmod q$. We let $A = \{0\}$ and take n_k to be the sequence of multiples of q. This has density $1/q$ and we also observe that $T^{-kq}A = 0$ for all $k \in N$, since $T^{kq}(0) = 0$. Therefore, we get that

$$\bigcup_{k=1}^{+\infty} T^{-kq}A = \{0\}$$

which has measure $1/q$. Thus, the rotation on q points is not good.

Now, let a be irrational and consider the rotation on the torus (with the Borel σ-algebra and the Lebesgue measure) by a. Namely, $Tx = x + a \pmod 1$. We let $A = [0, 1/4]$ and

$B = [1/2, 3/4]$ (any pair of disjoint intervals works here, with slight modifications on the choice of R below). Consider the set

$$R = \left\{ n \in \mathbb{N} \colon \{na\} \in \left[0, \frac{1}{100}\right] \right\},$$

where $\{na\} = na - \lfloor na \rfloor$. Since a is irrational, the sequence na is uniformly distributed (mod 1), which implies that the set R has positive density (as it is equal to $\mu([0, 1/100]) = 1/100$). We pick the sequence n_k to be the elements of the set R in increasing order. For any $x \in A$, we have that

$$T^{n_k} x = x + n_k a \pmod 1 \in \left[0, \frac{1}{4} + \frac{1}{100}\right].$$

Therefore, we have that $T^{-n_k} A \cap B = \emptyset$ which implies that

$$\mu\left(\bigcup_{k=1}^{+\infty} T^{-n_k} A\right) \leq \mu(\mathbb{T} \setminus B) \leq \frac{3}{4}.$$

We conclude that the rotation by a is not good.

To summarize, we have shown that if the original system is good but not weak-mixing, then it has a factor that is not good. This contradicts our claim, which implies that our original system is weak-mixing.

P2. **(a)** Let $(X, \mathcal{B}, \mu, T)$ be a weakly-mixing system. Show that for all $a \in (0, 1)$ and any $f \in L^\infty(X)$, we have that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n a} f(T^n x) = 0$$

for almost all $x \in X$.

Hint: See also exercise 1 in Week 4.

We split into two cases depending on whether a is irrational or not.

If a is irrational, we consider the system $(\mathbb{T}, \mathcal{B}(T), \lambda, R)$ to be the rotation by a on the torus and define the product $(X \times T, \mathcal{B} \times \mathcal{B}(\mathbb{T}), \mu \times \lambda, T \times R)$. Since a is irrational, the rotation by a is ergodic and the weak-mixing assumption implies that our product system is ergodic.

We let $g(y) = e(y), y \in \mathbb{T}$ and consider the function $f \otimes g(x, y) = f(x)g(y)$ defined on the product $X \times \mathbb{T}$. Using the pointwise ergodic theorem, we infer that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} (T \times R)^n (f \otimes g)(x, y) = \int_{X \times \mathbb{T}} f \otimes g \, d(\mu \times \lambda)$$

for $(\mu \times \lambda)$-almost all $(x, y) \in X \times \mathbb{T}$.

By Fubini's theorem, we have that

$$\int_{X \times \mathbb{T}} f \otimes g \, d(\mu \times \lambda) = \int_T f(x) \left(\int g(y) d\lambda(y) \right) d\mu(x) = 0,$$

since $g(y)$ has integral zero.

On the other hand, we have that

$$(T \times R)^n (f \otimes g)(x,y) = f(T^n x) g(S^n y) = f(T^n x) e(y+na) = e(y)e(na) f(T^n x).$$

Combining the last two calculations, we conclude that

$$\lim_{N \to +\infty} \frac{e(y)}{N} \sum_{n=0}^{N-1} e^{2\pi i n a} f(T^n x) = 0$$

for almost all $(x,y) \in (X \times \mathbb{T})$. Equivalently,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n a} f(T^n x) = 0$$

for $(\mu \times \lambda)$almost all $(x,y) \in (X \times \mathbb{T})$.
Let $A \subseteq X \times \mathbb{T}$ be the set of (x,y) for which the limit above is zero, so that $(\mu \times \lambda)(A) = 1$. Fubini's theorem implies that

$$\int_{\mathbb{T}} \left(\int_X 1_A(x,y) d\mu(x) \right) d\lambda(y) = 1.$$

Since $\int_X 1_A(x,y) d\mu(x) \leq 1$ for all $y \in \mathbb{T}$, there exists at least one $y_0 \in \mathbb{T}$ such that $\int_X 1_A(x,y_0) d\mu(x) = 1$. For this y_0, we have $1_A(x,y_0) = 1$ for almost all $x \in X$ (otherwise the integral would not equal 1), and, thus, for almost all $x \in X$, we have

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(2\pi i n a) f(T^n x) = 0,$$

which is the desired conclusion.
Now, assume that a is rational and write $a = p/q$ where p, q are coprime. Since the sequence $e(np/q)$ is periodic modulo q, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} e(\frac{np}{q}) f(T^n x) = \frac{1}{N} \sum_{0 \leq r \leq q-1} \sum_{\substack{0 \leq n \leq N-1 \\ n \equiv r \pmod q}} e(pr/q) f(T^n x) =$$

$$\sum_{0 \leq r \leq q-1} \frac{e(pr/q)}{N} \sum_{0 \leq n \leq \frac{N-1-r}{q}} f(T^{qn+r} x). \quad (3)$$

Since the system is weak-mixing, we know from Exercise 1 in Sheet 8 that T^q is weak-mixing and, thus, ergodic. Therefore, using the pointwise ergodic theorem (for the map T^q), we have that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{0 \leq n \leq \frac{N-1-r}{q}} f(T^{qn} x) = \lim_{N \to +\infty} \frac{\lfloor (N-1-r)/q \rfloor}{N} \sum_{0 \leq n \leq \frac{N-1-r}{q}} f(T^{qn} x) = \int f \, d\mu$$

for almost all $x \in X$. In fact, for a set of full measure A, we have that

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{0\le n\le \frac{N-1-r}{q}} f(T^{qn}x) = \int f\, d\mu$$

for all $0 \le r \le q-1$.
Since $\mu(A) = 1$, we have that $\mu(A \cap T^{-1}A \cap \cdots \cap T^{-q+1}A) = 1$. Then, for every point $x \in A \cap T^{-1}A \cap \cdots \cap T^{-q+1}A$, we have that $T^r x \in A$ for every $0 \le r \le q-1$. We conclude that for almost all $x \in X$, we have

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{0\le n\le \frac{N-1-r}{q}} f\left(T^{qn}(T^r x)\right) = \int f\, d\mu$$

for every $0 \le r \le q-1$.
For all x satisfying the conclusion above, we have that the limit in (3) is equal to

$$\sum_{0\le r\le q-1} e(pr/q) \int f\, d\mu = \int f\, d\mu \left(\sum_{0\le r\le q-1} e(pr/q) \right).$$

Since p and q are coprime, the numbers $pr \pmod q$ take each value in the residue classes modulo q exactly once. Therefore, the previous sum is equal to the sum over all the q-roots of unity and, hence, they add up to zero. The result follows.

(b) Let $(X, \mathcal{B}, \mu, T)$ be a mixing system. Show that for all $f \in L^\infty(X)$, we have

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} f - \int f d\mu \right\|_{L^2(X)} = 0.$$

Hint: $|2^n - 2^m|$ is "large" for "almost all" pairs (n, m).

We set $g = f - \int f\, d\mu$ (so that $\int g\, d\mu = 0$) and we rewrite our expression as

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} g \right\|_{L^2(X)} = 0.$$

Rescaling g, we can assume that $\|g\|_{L^\infty(X)} \le 1$.
We have that

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} g \right\|_{L^2(\mu)} = \int \left| \frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} g \right|^2 d\mu = \int \left(\frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} g \right) \overline{\left(\frac{1}{N} \sum_{n=0}^{N-1} T^{2^n} g \right)}\, d\mu =$$
$$\int \frac{1}{N^2} \sum_{0\le n,m\le N-1} T^{2^n} g \cdot \overline{T^{2^m} g}\, d\mu = \frac{1}{N^2} \sum_{0\le n,m\le N-1} \int T^{2^n} g \cdot \overline{T^{2^m} g}\, d\mu.$$

We want to show that this last double average converges to zero, as $N \to +\infty$.
Since the system is mixing, we have that

$$\lim_{n\to+\infty} \int \overline{g} \cdot T^n g\, d\mu = 0.$$

Let $\varepsilon > 0$. Then, there exists R such that if $n \geq R$, we have

$$\left| \int \overline{g} \cdot T^n g \, d\mu \right| < \varepsilon.$$

We split our average into three sums depending on whether $n > m$, $n < m$ or $n = m$. For the diagonal contribution, we just bound trivially

$$\left| \frac{1}{N^2} \sum_{n=0}^{N-1} \int T^{2^n} g \cdot \overline{T^{2^m}} g \, d\mu \right| \leq \frac{1}{N^2} \sum_{n=0}^{N-1} \left| \int T^{2^n} g \cdot \overline{T^{2^m}} g \, d\mu \right| \leq \frac{1}{N^2} \sum_{n=0}^{N-1} 1 = \frac{1}{N},$$

since g is 1-bounded.
We now bound the absolute value of the sum

$$\frac{1}{N^2} \sum_{0 \leq m < n \leq N-1} \int T^{2^n} g \cdot \overline{T^{2^m}} g \, d\mu.$$

Since T is measure-preserving, we can rewrite this as

$$\frac{1}{N^2} \sum_{0 \leq m < n \leq N-1} \int \overline{g} \cdot T^{2^n - 2^m} g \, d\mu.$$

We count how many pairs (n, m) satisfy the inequality $|2^n - 2^m| < R$. Observe that $R > |2^m(2^{n-m} - 1)|$ implies that both $m < \log_2 R$ and $n - m < \log_2(R + 1)$, from which we get the condition that $m < \log_2 R$ and $n < \log_2((R + 1)R)$. Thus, there are at most $\log_2(R) \cdot \log_2(R(R + 1)) < 2 \log_2^2(R + 1)$ pairs (n, m) for which $|2^n - 2^m| < R$. For all other pairs, we have that $\left| \int \overline{g} \cdot T^{2^n - 2^m} g \, d\mu \right| < \varepsilon$.
We let S be the pairs (n, m) of integers for which $2^n - 2^m < R$. Then, for N much larger than R, we have

$$\left| \frac{1}{N^2} \sum_{0 \leq m < n \leq N-1} \int \overline{g} \cdot T^{2^n - 2^m} g \, d\mu \right| \leq \frac{1}{N^2} \sum_{0 \leq m < n \leq N-1} \left| \int \overline{g} \cdot T^{2^n - 2^m} g \, d\mu \right| =$$
$$\frac{1}{N^2} \sum_{\substack{0 \leq m < n \leq N-1 \\ (n,m) \in S}} \left| \int \overline{g} \cdot T^{2^n - 2^m} g \, d\mu \right| + \frac{1}{N^2} \sum_{\substack{0 \leq m < n \leq N-1 \\ (n,m) \notin S}} \left| \int \overline{g} \cdot T^{2^n - 2^m} g \, d\mu \right| \leq$$
$$\frac{1}{N^2} \sum_{\substack{0 \leq m < n \leq N-1 \\ (n,m) \in S}} 1 + \frac{1}{N^2} \sum_{\substack{0 \leq m < n \leq N-1 \\ (n,m) \notin S}} \varepsilon = \frac{|\{(n, m) \colon 0 \leq m < n \leq N - 1, (n, m) \in R\}|}{N^2} +$$
$$\frac{|\{(n, m) \colon 0 \leq m < n \leq N - 1, (n, m) \notin R\}|}{N^2} \varepsilon < \frac{2 \log_2^2(R + 1)}{N^2} + \frac{N^2}{N^2} \varepsilon = \varepsilon + \frac{2 \log_2^2(R + 1)}{N^2}.$$

An entirely similar argument gives the bound

$$\left| \frac{1}{N^2} \sum_{0 \leq n < m \leq N-1} \int \overline{T^{2^m} g} \cdot T^{2^n} g \, d\mu \right| < \varepsilon + \frac{2 \log_2^2(R + 1)}{N^2}.$$

If we combine the 3 bounds, we get that

$$\left|\frac{1}{N^2}\sum_{0\leq n,m\leq N-1}\int T^{2^n}g\cdot\overline{T^{2^m}g}\,d\mu.\right|\leq 2\varepsilon+\frac{4\log_2^2(R+1)}{N^2}+\frac{1}{N}.$$

Sending $N\to+\infty$, we infer that

$$\limsup_{N\to+\infty}\left\|\frac{1}{N}\sum_{n=0}^{N-1}T^{2^n}g\right\|_{L^2(X)}\leq 2\varepsilon.$$

Since ε was arbitrary, the result follows.

P3. A measure preserving system $(X,\mathcal{B},\mu,T)$ is called *rigid* if there is an increasing sequence $(n_k)_{k\in\mathbb{N}}\subseteq\mathbb{N}$ such that for all $f\in L^2(X,\mathcal{B},\mu)$, we have $\|f\circ T^{n_k}-f\|_2\to 0$ as $k\to\infty$.

(a) Prove that $(X,\mathcal{B},\mu,T)$ is rigid if and only if there is an increasing sequence $(n_k)_{k\in\mathbb{N}}\subseteq\mathbb{N}$, a dense subset $V\subseteq L^2(X,\mathcal{B},\mu)$ such that for all $f\in V$, $\|f\circ T^{n_k}-f\|_2\to 0$ as $k\to\infty$.

The first direction is immediate by picking V to be any dense subset of $L^2(X)$ (for example, the set of all linear combinations of characteristic functions of measurable sets).
For the other direction, let V be a dense subset of $L^2(X)$ and let f be any square integrable function. Let $\varepsilon>0$ and pick a function $g\in V$ such that $\|f-g\|_{L^2(X)}\leq\varepsilon$. Then, we have

$$\|T^{n_k}f-f\|_{L^2(X)}\leq\|T^{n_k}f-T^{n_k}g\|_{L^2(X)}+\|T^{n_k}g-g\|_{L^2(X)}+\|g-f\|_{L^2(x)}=$$
$$2\|f-g\|_{L^2(X)}+\|T^{n_k}g-g\|_{L^2(X)}\leq 2\varepsilon+\|T^{n_k}g-g\|_{L^2(X)},$$

where we used the fact that T preserves μ. Since $g\in V$, we have $\|T^{n_k}g-g\|_{L^2(X)}\to 0$ and, thus,

$$\limsup_{k\to+\infty}\|T^{n_k}f-f\|_{L^2(X)}\leq 2\varepsilon.$$

Since ε was arbitrary, the result follows.

(b) Suppose that $(X,\mu,\mathcal{B},T)$ has discrete spectrum. Prove that $(X,\mu,\mathcal{B},T)$ is rigid.
Hint: Use part **(a)** for a suitable dense subspace $V\subseteq L^2(X,\mathcal{B},\mu)$, and also use **P2** from the Exercise sheet 6.

Solution: As the system has discrete spectrum, we have that there is an orthonormal basis $\{f_k\}_{k\in\mathbb{N}}$ of $L^2(X)$ consisting of eigenfunctions. Thus, it is enough to prove the statement for $V=\langle\{f_k\}_{k\in\mathbb{N}}\rangle$ by the previous part.
For every $k\in\mathbb{N}$, denote $\alpha_k\in\mathbb{T}$ the element such that $e^{2\pi i\alpha_k}$ is eigenvalue associated to f_k. Let $(n_k)_k$ be the sequence given by Problem 2 from Exercise sheet 6. We claim that this sequence satisfies the desired property. Indeed, let $K\in\mathbb{K}$ and $a_1,\ldots,a_K\in\mathbb{R}$. Then

consider $f = \sum_{j=1}^{K} a_j f_j$. We have that for every $k \geq K$

$$\begin{aligned}
\|f \circ T^{n_k} - f\|_2 &\leq \sum_{j=1}^{K} \|a_j f_j \circ T^{n_k} - a_j f_j\|_2 \\
&= \sum_{j=1}^{K} |a_j| |e^{2\pi i n_k \alpha_j} - 1| \underbrace{\|f_j\|_2}_{=1} \\
&\leq \sum_{j=1}^{K} |a_j| \|n_k \alpha_j\|_{\mathbb{T}} \\
&\leq \frac{1}{k} K \max_{j=1,\ldots,K} |a_j|.
\end{aligned}$$

Taking $k \to \infty$ we get $\|f \circ T^{n_k} - f\|_2 \to 0$. Thus, any function $f \in V$ satisfies the desired property and since V is dense in $L^2(X)$, we get the desired conclusion.

(c) We call a system $(X, \mathcal{B}, \mu, T)$ mildly mixing if it has no non-trivial rigid factors. Namely, there does not exist a factor map $(X, , \mu, T) \to (Y, \mathcal{A}, \nu, S)$ such that the system $(Y, \mathcal{A}, \nu, S)$ is rigid.

Show that a mixing system is mildly mixing.

First of all, we observe that a factor of mixing system is also mixing (to see this, argue similarly as in Exercise 1b) in Sheet 8). Therefore, our result will follow if we show that there cannot be a non-trivial system $(X, \mathcal{B}, \mu, T)$ that is both rigid and mixing.

Indeed, assume that the system is both mixing and rigid. Then, there exists a sequence $n_k \to +\infty$ such that $\|T^{n_k} f - f\|_{L^2(X)} \to 0$ for all $f \in L^2(X)$. Using the Cauchy-Schwarz inequality, we deduce that

$$\left| \int \overline{f} \cdot T^{n_k} f \, d\mu - \int |f|^2 \, d\mu \right| \leq \int \left| \overline{f} \cdot (T^{n_k} f - f) \right| \, d\mu \leq \|f\|_{L^2(X)} \|T^{n_k} f - f\|_{L^2(X)} \to 0.$$

Since the system is mixing, we have that

$$\lim_{k \to +\infty} \int \overline{f} \cdot T^{n_k} f \, d\mu \to \int \overline{f} \, d\mu \int f \, d\mu = \left| \int f \, d\mu \right|^2.$$

Combining the above, we conclude that for any $f \in L^2(X)$, we have

$$\int |f|^2 \, d\mu = \left| \int f \, d\mu \right|^2$$

Therefore, f is almost everywhere equal to a constant (it satisfies the equality case in the Cauchy-Schwarz inequality) and thus the system is isomorphic to the trivial system. The result follows.

P4. Consider the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), \mu, T)$ where μ is the Haar measure in $\mathbb{T}^2$ and T is the baker's map defined by

$$T(x, y) = (2x - \lfloor 2x \rfloor, \frac{y + \lfloor 2x \rfloor}{2}).$$

Prove that this map is a Bernoulli system.

Solution: Consider the map $\varphi : (\{0,1\}^{\mathbb{Z}}, \nu, S) \to (\mathbb{T}^2, \mu, T)$ given by

$$\varphi(\sigma) = (\sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)}, \sum_{k=0}^{\infty} \sigma_{k+1} 2^{-(k+1)}).$$

for every sequence $\sigma = (\sigma_k)_{k\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$. We see that this map is an isomorphism. Indeed, this map represents the 2-ary representations of any number on $\mathbb{T}^2$, and therefore this map is a bijection in a set of full measure (for example, in

$$X_0 = \{0,1\}^{\mathbb{Z}} \setminus \{\sigma \in \{0,1\}^{\mathbb{Z}} \colon \exists K \in \mathbb{N}, (\sigma_k = 1, \forall k \geq K) \vee (\sigma)_{-k} = 1, \forall k \geq K)\}).$$

On the other hand, notice that for $\sigma \in \{0,1\}^{\mathbb{Z}}$,

$$\begin{aligned}
\varphi(S\sigma) &= (\sum_{k=0}^{\infty} \sigma_{-k-1} 2^{-(k+1)}, \sum_{k=0}^{\infty} \sigma_k 2^{-(k+1)}) \\
&= (2\sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)} - \sigma_0, \frac{\sum_{k=0}^{\infty} \sigma_{k+1} 2^{-(k+1)} + \sigma_0}{2}) \\
&= (2\sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)} - \lfloor 2\sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)} \rfloor, \frac{\sum_{k=0}^{\infty} \sigma_{k+1} 2^{-(k+1)} + \lfloor 2\sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)} \rfloor}{2}) \\
&= T(\varphi(\sigma)).
\end{aligned}$$

Finally, we see that $\varphi\nu = \mu$. Indeed, notice that it is enough to prove it for a basic open set of the form $[\frac{j}{2^k}, \frac{j+1}{2^k}) \times [\frac{i}{2^l}, \frac{i+1}{2^l}) \in \mathcal{B}(\mathbb{T}^2)$ for $j < 2^k$ and $i < 2^l$, since these generate the product σ-algebra. We have that

$$\varphi\nu([\frac{j}{2^k}, \frac{j+1}{2^k}) \times [\frac{i}{2^l}, \frac{i+1}{2^l})) =$$
$$\nu(\{\sigma \in \{0,1\}^{\mathbb{Z}} \colon \sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)} \in [\frac{j}{2^k}, \frac{j+1}{2^k}), \ \sum_{k=0}^{\infty} \sigma_{k+1} 2^{-(k+1)} \in [\frac{i}{2^l}, \frac{i+1}{2^l})\}).$$

Observe that $\sum_{k=0}^{\infty} \sigma_{-k} 2^{-(k+1)} \in [\frac{j}{2^k}, \frac{j+1}{2^k})$ fixes k coordinates, meanwhile $\sum_{k=0}^{\infty} \sigma_{k+1} 2^{-(k+1)} \in [\frac{i}{2^l}, \frac{i+1}{2^l})$ fixes l different coordinates. Therefore, we have

$$\varphi\nu([\frac{j}{2^k}, \frac{j+1}{2^k}) \times [\frac{i}{2^l}, \frac{i+1}{2^l}))) = \frac{1}{2^k} \cdot \frac{1}{2^l} = \mu([\frac{j}{2^k}, \frac{j+1}{2^k}) \times [\frac{i}{2^l}, \frac{i+1}{2^l})),$$

and the result follows.